

EXISTENCE OF APPROXIMATE HERMITIAN-EINSTEIN STRUCTURES ON SEMI-STABLE HIGGS BUNDLES

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ABSTRACT. In this paper, using Donaldson's heat flow, we show that the semi-stability of a Higgs bundle over a compact Kähler manifold implies the existence of approximate Hermitian-Einstein structure on the Higgs bundle.

1. INTRODUCTION

Let (M, ω) be a compact Kähler manifold, and E be a holomorphic vector bundle over M . The stability of holomorphic vector bundles, in the sense of Mumford-Takemoto, was a well established concept in algebraic geometry. A holomorphic vector bundle E is called stable (semi-stable), if for every coherent sub-sheaf $E' \hookrightarrow E$ of lower rank, it holds:

$$(1.1) \quad \mu(E') = \frac{\deg(E')}{\text{rank } E'} < (\leq) \mu(E) = \frac{\deg(E)}{\text{rank } E},$$

where $\mu(E')$ is called the slope of E' . In early 1980s, S. Kobayashi introduced the Hermitian-Einstein condition for holomorphic bundles on Kähler manifolds. A Hermitian metric H in E is said to be Hermitian-Einstein, if the curvature F_H of the Chern connection D_H satisfies the Einstein condition:

$$(1.2) \quad \sqrt{-1} \Lambda_\omega F_H = \lambda Id_E,$$

where Λ_ω denotes the contraction of differential forms by Kähler form ω , and the real constant λ is given by $\lambda = \frac{2\pi}{\text{Vol}(M)} \mu(E)$.

The so-called Hitchin-Kobayashi correspondence asserts that holomorphic vector bundles over compact Kähler manifolds are polystable if and only if they admit a Hermitian-Einstein metric. This correspondence starts by Narasimhan and Seshadri ([21]) in the case of compact Riemannian surface. Kobayashi ([14]) and Lübke ([19]) proved that a holomorphic bundle admits a Hermitian-Einstein metric must be polystable. The inverse problem was solved by Donaldson ([9], [10]) for algebraic manifolds, by Uhlenbeck and Yau ([24]) for general Kähler manifolds. Donaldson-Uhlenbeck-Yau theorem states that the stability of a holomorphic vector bundle implies the existence of Hermitian-Einstein metric. The classical Hitchin-Kobayashi correspondence has several interesting and important generalizations and extensions where some extra structures are added to the holomorphic bundles, see references: [12], [22],[3],[11], [4], [1], [2], [6], [16], [17], [18], [20], [23].

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A holomorphic bundle $(E, \bar{\partial}_E)$ coupled with one Higgs field $\theta \in \Omega^{1,0}(End(E))$ which satisfying $\bar{\partial}_E \theta = 0$ and $\theta \wedge \theta = 0$ will be called by a Higgs bundle. A Higgs bundle (E, θ) is called Stable (Semi-stable) if the usual stability condition $\mu(E') < \mu(E)$ (\leq) hold for all proper θ -invariant sub-sheaves. A Hermitian metric H in Higgs bundle (E, θ) is said to be Hermitian-Einstein if the curvature F of the Hitchin-Simpson connection $D_{H,\theta} = D_H + \theta + \theta^{*H}$ satisfies the Einstein condition, i.e

$$(1.3) \quad \sqrt{-1}\Lambda_\omega(F_H + [\theta, \theta^{*H}]) = \lambda Id_E,$$

where F_H is the curvature of Chern connection D_H , θ^{*H} is the adjoint of θ with respect to the metric H .

Higgs bundles first emerged twenty years ago in Hitchin's study of the self-duality equations on a Riemann surface and in Simpsons work on nonabelian Hodge theory. Higgs bundles have a rich structure and play a role in many different areas including gauge theory, Kähler and hyperkähler geometry, group representations and nonabelian Hodge theory. In [12] and [22], it is proved that a Higgs bundle admits the Hermitian-Einstein metric iff it's Higgs poly-stable. This is a Higgs bundle version of the Donaldson-Uhlenbeck-Yau theorem.

We say a holomorphic vector bundle E admits an approximate Hermitian-Einstein structure if for every positive ϵ , there is a Hermitian metric H such that

$$(1.4) \quad \max_M |\sqrt{-1}\Lambda_\omega F_H - \lambda Id_E|_H < \epsilon.$$

Kobayashi ([15]) introduced the notion of approximate Hermitian-Einstein structure in a holomorphic vector bundle, and he proved that a holomorphic vector bundle with an approximate Hermitian-Einstein structure must be semi-stable. Furthermore, over projective algebraic manifolds, Kobayashi solved the inverse problem, i.e. the semi-stability implies admitting an approximate Hermitian-Einstein structure, and he also conjectured that the result should be true for general compact Kähler manifolds.

In this article, we consider the existence problem of approximate Hermitian-Einstein structure in Higgs bundles. We will show that the semi-stability of Higgs bundle implies the existence of approximate Hermitian-Einstein structures. In fact, we prove the following theorem.

Theorem 1. *If (E, θ) is a semi-stable Higgs bundle on Kähler manifold (M, ω) , then it admits an approximate Hermitian-Einstein structure, i.e. for any $\epsilon > 0$ there exists a Hermitian metric such that*

$$(1.5) \quad \max_M |\sqrt{-1}\Lambda_\omega(F_H + [\theta, \theta^{*H}]) - \lambda Id_E|_H < \epsilon.$$

We will use the heat flow method to prove theorem 1. Simpson ([22]) introduced the Donaldson's heat flow to Higgs bundle case, i.e. the following heat flow for Hermitian metrics on the Higgs bundle (E, θ) with initial metric H_0 :

$$(1.6) \quad H^{-1} \frac{\partial H}{\partial t} = -2(\sqrt{-1}\Lambda_\omega(F_H + [\theta, \theta^{*H}]) - \lambda Id_E).$$

Simpson proved the long time existence and uniqueness of solution for the above non-linear heat equation. By Uhlenbeck and Yau's result, that L_1^2 weakly holomorphic sub-bundle defines a coherent sub-sheaf, Simpson obtain an uniform C^0 -estimate of the long time solution of the above heat flow (1.6) when the Higgs bundle is stable, and show that the solution must convergence to a Hermitian-Einstein metric. In this article, we will follow Simpson's discussion to prove that, along the heat flow, the term $\max_M |\sqrt{-1}\Lambda_\omega(F_H + [\theta, \theta^{*H}]) - \lambda Id|_H$ must convergence to zero under the assumption that the Higgs bundle is semi-stable. The correspondence between semistability and the existence of approximate Hermitian-Einstein structure in the holomorphic vector bundle case has been studied recently by Jacob ([13]) using a technique developed by Buchdahl ([7]) for the regularization of sheaves in the case of compact complex surfaces. It should be point out that our discussion is different from that in [13]. Recently, Cardona ([8]) obtain the result of theorem 1 in Riemann surface case by using a Donaldson functional approach analogous to that of Kobayashi [15].

In [5], Bruzzo and Otero proved that a Higgs bundle admitting an approximate Hermitian-Einstein structure must be semi-stable. Combining Bruzzo and Otero's result and theorem 1, we know that, in Higgs bundles, admitting an approximate Hermitian-Einstein structure and the semi-stability are equivalent. It is easy to check that if two Higgs bundles admit approximate Hermitian-Einstein structure, so does their tensor product; furthermore, if they are with the same slope, so does their Whitney sum. So, we have the following corollary.

Corollary 2. *Let (E_1, θ_1) and (E_2, θ_2) be two semi-stable Higgs bundle on Kähler manifold (M, ω) , then*

- (1), $(E_1 \otimes E_2, \theta)$ is a semi-stable Higgs bundle, where $\theta = \theta_1 \otimes Id_2 + Id_1 \otimes \theta_2$;
- (2), If $\frac{\deg(E_1)}{\text{rank } E_1} = \frac{\deg(E_2)}{\text{rank } E_2}$, then $(E_1 \oplus E_2, \theta)$ is also a semi-stable Higgs bundle, where $\theta = pr_1^* \theta_1 + pr_2^* \theta_2$ and $pr_i : E_1 \oplus E_2 \rightarrow E_i$ denote the natural projections.

Another application of theorem 1 is the following Bogomolov type inequality for semi-stable Higgs bundle. By Chern-Weil theory, we have

$$\begin{aligned}
 & 4\pi^2 \int_M (2C_2(E) - \frac{r-1}{r} C_1(E) \wedge C_1(E)) \frac{\omega^{n-2}}{(n-2)!} \\
 (1.7) \quad &= \int_M \text{tr}(F_{H,\theta}^\perp \wedge F_{H,\theta}^\perp) \wedge \frac{\omega^{n-2}}{(n-2)!} \\
 &= \int_M |F_{H,\theta}^\perp|_H^2 - |\Lambda_\omega F_{H,\theta}^\perp|_H^2 \frac{\omega^n}{n!} \\
 &\geq - \int_M |\sqrt{-1}\Lambda_\omega F_{H,\theta} - \lambda Id - \frac{1}{r} \text{tr}(\sqrt{-1}\Lambda_\omega F_{H,\theta} - \lambda Id) Id|_H^2 \frac{\omega^n}{n!},
 \end{aligned}$$

where $F_{H,\theta}^\perp$ is the trace free part of $F_{H,\theta}$. If the Higgs bundle (E, θ) admits an approximate Hermitian-Einstein structure, we can choose a sequence of metric H_i so that the last term of the above inequality converges to zero, so we obtain the following corollary.

Corollary 3. *If (E, θ) is a semi-stable Higgs bundle on Kähler manifold (M, ω) , then we have the following Bogomolov type inequality*

$$(1.8) \quad \int_M (2C_2(E) - \frac{r-1}{r} C_1(E) \wedge C_1(E)) \frac{\omega^{n-2}}{(n-2)!} \geq 0.$$

This paper is organized as follows. In Section 2, we recall some basic estimates for the Donaldson's heat flow in Higgs bundle. In section 3, we prove theorem 1.

2. ANALYTIC PRELIMINARIES AND BASIC ESTIMATES

Suppose $H(t)$ is a solution of the above Donaldson's heat flow (1.6) with initial metric K , and let $h(t) = K^{-1}H(t)$, then (1.6) can be written as

$$(2.1) \quad \frac{\partial h}{\partial t} = -2\sqrt{-1}h\Lambda_\omega(F_K + \bar{\partial}_E(h^{-1}\partial_K h) + [\theta, h^{-1}\theta^{*K}h]) + 2\lambda h.$$

Furthermore, by an appropriate conformal change, we can assume that the initial metric K satisfies

$$(2.2) \quad \text{tr}(\sqrt{-1}\Lambda_\omega(F_K + [\theta, \theta^{*K}]) - \lambda Id_E) = 0.$$

In fact, set $K = e^f H_0$ and f is defined by the Poisson equation

$$\Delta f = \frac{2}{r} \text{tr}(\sqrt{-1}\Lambda_\omega(F_{H_0} + [\theta, \theta^{*H_0}]) - \lambda Id),$$

by noting that $\int_M \text{tr}(\sqrt{-1}\Lambda_\omega(F_{H_0} + [\theta, \theta^{*H_0}]) - \lambda Id) \frac{\omega^m}{m!} = 0$.

For simplicity, we denote:

$$(2.3) \quad \Phi(H, \theta) = \sqrt{-1}\Lambda_\omega(F_H + [\theta, \theta^{*H}]) - \lambda Id_E.$$

It is easy to check that $(\sqrt{-1}\Lambda_\omega F_H)^{*H} = \sqrt{-1}\Lambda_\omega F_H$ and $(\Phi(H, \theta))^{*H} = \Phi(H, \theta)$. The following lemma is essentially proved by Simpson ([22]), we give a proof just for completeness.

Lemma 4. *Let $H(t)$ be a solution of the heat flow (1.6) with initial metric K , then we have:*

$$(2.4) \quad \left(\frac{\partial}{\partial t} - \Delta\right) \text{tr}(\Phi(H, \theta)) = 0$$

and

$$(2.5) \quad \left(\frac{\partial}{\partial t} - \Delta\right) |\Phi(H, \theta)|_H^2 = -4|D_\theta'' \Phi(H, \theta)|_H^2,$$

where $D_\theta'' = \bar{\partial}_E + \theta$.

Proof. Using the identities

$$(2.6) \quad \begin{aligned} \partial_H - \partial_K &= h^{-1}\partial_K h; \\ F_H - F_K &= \bar{\partial}_E(h^{-1}\partial_K h); \\ \theta^{*H} &= h^{-1}\theta^{*K}h, \end{aligned}$$

we have:

$$(2.7) \quad \begin{aligned} \frac{\partial}{\partial t} \Phi(H(t), \theta) &= \sqrt{-1}\Lambda_\omega\{\bar{\partial}_E(-h^{-1}\frac{\partial h}{\partial t}h^{-1}\partial_K h + h^{-1}\partial_K(\frac{\partial h}{\partial t})) \\ &\quad - [\theta, h^{-1}\frac{\partial h}{\partial t}h^{-1}\theta^{*K}h] + [\theta, h^{-1}\theta^{*K}\frac{\partial h}{\partial t}]\} \\ &= \sqrt{-1}\Lambda_\omega\{\bar{\partial}_E(\partial_H(h^{-1}\frac{\partial h}{\partial t})) + [\theta, [\theta^{*H}, h^{-1}\frac{\partial h}{\partial t}]]\}. \end{aligned}$$

The formula (2.4) can be deduced from (2.7) directly. On the other hand,

$$(2.8) \quad \begin{aligned} \Delta |\Phi|_H^2 &= -2\sqrt{-1}\Lambda_\omega \bar{\partial} \text{tr}(\Phi H^{-1} \bar{\Phi}^t H) \\ &= -2\sqrt{-1}\Lambda_\omega \bar{\partial} \text{tr}\{\partial \Phi H^{-1} \bar{\Phi}^t H - \Phi H^{-1} \partial H H^{-1} \bar{\Phi}^t H \\ &\quad + \Phi H^{-1} \bar{\partial} \bar{\Phi}^t H + \Phi H^{-1} \bar{\Phi}^t H H^{-1} \partial H \\ &= 2\text{Re} < -2\sqrt{-1}\Lambda_\omega \bar{\partial}_E \partial_E \Phi, \Phi >_H + 2|\partial_H \Phi|_H^2 + 2|\bar{\partial}_E \Phi|_H^2 \\ &\quad - 2\sqrt{-1}\Lambda_\omega \text{tr}(\Phi H^{-1} [\bar{F}_H, \Phi] H). \end{aligned}$$

By (2.7) and (2.9), we have

$$\begin{aligned}
 (2.9) \quad & (\Delta - \frac{\partial}{\partial t})|\Phi|_H^2 = \Delta|\Phi|_H^2 - 2Re < \frac{\partial}{\partial t}\Phi, \Phi >_H \\
 & = 4Re < \sqrt{-1}\lambda_\omega[\theta, [\theta^{*H}, \Phi], \Phi >_H + 2|\partial_H\Phi|_H^2 + 2|\bar{\partial}_E\Phi|_H^2 \\
 & = 2|\partial_H\Phi + [\theta^{*H}, \Phi]|_H^2 + 2|\bar{\partial}_E\Phi + [\theta, \Phi]|_H^2.
 \end{aligned}$$

Since $\Phi(H, \theta)$ is self adjoint with respect to H , (2.9) implies (2.5). \square

Set $h = K^{-1}H = \exp(S)$, where $S \in \text{End}(E)$ and it is self adjoint with respect to K or H . By the initial condition and (2.4), we have $\text{tr}(\Phi(H(t), \theta)) = 0$, then $\frac{\partial}{\partial t} \det(h(t)) = \text{tr}(h^{-1} \frac{\partial h}{\partial t}) = 0$. So we have

$$(2.10) \quad \det(h(t)) = 1,$$

and

$$(2.11) \quad \text{tr}(S(t)) = 0.$$

By (2.5) and the maximum principle, we know that $\|\Phi(H(t), \theta)\|_{L^\infty}$ and $\|\Phi(H(t), \theta)\|_{L^2}$ are monotonely decreasing.

For reader's convenience, we recall some notation. Let K be a fixed Hermitian metric on bundle E , denote

$$(2.12) \quad S_K(E) = \{\eta \in \Omega^0(M, \text{End}(E)) \mid \eta^{*K} = \eta\}.$$

Given $\rho \in C^\infty(R, R)$ and $\eta \in S_K(E)$. We define $\rho(\eta)$ as follows. At each point x on M , choose an unitary basis $\{e_i\}_{i=1}^r$ with respect to metric K , such that $\eta(e_i) = \lambda_i e_i$. Set:

$$(2.13) \quad \rho(\eta)(e_i) = \rho(\lambda_i) e_i.$$

Given $\Psi \in C^\infty(R \times R, R)$, $\eta \in S_K(E)$, $p \in \Omega^0(M, \text{End}(E))$. In a similar way, we define $\Psi[\eta](p)$ as follows. Let $\{e_i^*\}_{i=1}^r$ be the dual basis for $\{e_i\}_{i=1}^r$, then $p \in \Omega^0(M, \text{End}(E))$ can be written

$$p = \sum p_{ij} e_i^* \otimes e_j.$$

Setting

$$(2.14) \quad \Psi[\eta](p) = \sum \Psi(\lambda_i, \lambda_j) p_{ij} e_i^* \otimes e_j.$$

Let's recall Donaldson's functional defined on the space of Hermitian metrics on Higgs bundle (E, θ) (Simpson [22]),

$$(2.15) \quad \mu(K, H) = \int_M \text{tr}(S \sqrt{-1} \Lambda_\omega F_{K, \theta}) + < \Psi(S)(D_\theta'' S), D_\theta'' S >_K \frac{\omega^n}{n!},$$

where $\Psi(x, y) = (x - y)^{-2}(e^{y-x} - (y - x) - 1)$, $\exp S = K^{-1}H$. The Donaldson's functional has another equivalent form, i.e.

$$(2.16) \quad \mu(K, H) = \int_0^1 \int_M \text{tr}(\Phi(H(s), \theta) H(s)^{-1} \frac{\partial H}{\partial s}) \frac{\omega^n}{n!} ds,$$

where $H(s)$ is any path connecting metrics K and H . The above integral is independent on path, and we also have a formula for the derivative with respect to t of Donaldson's functional,

$$(2.17) \quad \frac{d}{dt} \mu(K, H(t)) = \int_M \text{tr}(\Phi(H(s), \theta) H(t)^{-1} \frac{\partial H}{\partial t}) \frac{\omega^n}{n!},$$

for details see recent paper [8]. By (2.17), we know that the heat flow (1.6) is the gradient flow of Donaldson's functional.

3. PROOF OF THEOREM 1

Let $H(t)$ be the solution of the heat flow (1.6) with initial metric K , then we have

$$(3.1) \quad \frac{d}{dt}\mu(K, H(t)) = -2 \int_M |\Phi(H(t), \theta)|_{H(t)}^2 \frac{\omega^n}{n!},$$

and

$$(3.2) \quad -\mu(K, H(t)) = 2 \int_0^t \int_M |\Phi(H(s), \theta)|_{H(s)}^2 \frac{\omega^n}{n!} ds.$$

Case 1, $\mu(K, H(t)) \geq -C > -\infty$, i.e.

$$(3.3) \quad \int_0^{+\infty} \int_M |\Phi(H(t), \theta)|_{H(t)}^2 \frac{\omega^n}{n!} dt < \infty.$$

By the monotonicity of the integral, we have

$$(3.4) \quad \int_M |\Phi(H(t), \theta)|_{H(t)}^2 \frac{\omega^n}{n!} \rightarrow 0$$

as $t \rightarrow +\infty$.

Case 2, $\mu(K, H(t)) \rightarrow -\infty$.

By the definition of Donaldson's functional (2.15), we have

$$(3.5) \quad \mu(K, H(t)) \geq -C \|S(t)\|_{L^\infty}$$

where $\exp(S(t)) = h(t) = K^{-1}H(t)$. From Lemma 4, we know that $\max_M |\sqrt{-1}\Lambda_\omega F_{H(t), \theta}|$ are uniformly bounded, so we have the following Simpson's estimate (p885 in [22])

$$(3.6) \quad \|S(t)\|_{L^\infty} \leq C_1 \|S(t)\|_{L^1} + C_2,$$

where constants C_1 and C_2 depend only on the curvature of initial metric K and the geometry of (M, ω) . Then, (3.5) implies

$$(3.7) \quad \|S(t)\|_{L^1} \rightarrow \infty$$

as $t \rightarrow \infty$. By direct calculation, we have

$$(3.8) \quad \begin{aligned} & \frac{\partial}{\partial t} \log(trh(t) + trh^{-1}(t)) \\ &= \frac{tr(hh^{-1} \frac{\partial h}{\partial t}) - tr(h^{-1} \frac{\partial h}{\partial t} h^{-1})}{trh + trh^{-1}} \\ &\leq |\Phi(H(t), \theta)|_{H(t)}, \end{aligned}$$

and

$$(3.9) \quad \log\left(\frac{1}{2r}(trh + trh^{-1})\right) \leq |S| \leq r^{\frac{1}{2}} \log(trh + trh^{-1}),$$

where $r = \text{rank}(E)$. By the above two inequalities, we have

$$(3.10) \quad \begin{aligned} & r^{-\frac{1}{2}} \|S(t)\|_{L^1} - V \log 2r \\ &\leq \int_M \log(trh(t) + trh^{-1}(t)) \frac{\omega^n}{n!} - V \log 2r \\ &\leq \int_0^t \|\Phi(H(s), \theta)\|_{L^1} ds \\ &\leq t^{\frac{1}{2}} \left(\int_0^t \|\Phi(H(s), \theta)\|_{L^1}^2 ds \right)^{\frac{1}{2}} \\ &\leq V^{\frac{1}{2}} t^{\frac{1}{2}} \left(\int_0^t \|\Phi(H(s), \theta)\|_{L^2}^2 ds \right)^{\frac{1}{2}} \\ &\leq (Vt)^{\frac{1}{2}} (-\mu(K, H(t)))^{\frac{1}{2}}, \end{aligned}$$

where V is the volume of (M, ω) . On the other hand, the monotonicity of $\|\Phi(H(s), \theta)\|_{L^2}$ implies

$$(3.11) \quad \begin{aligned} & t\|\Phi(H(t), \theta)\|_{L^2}^2 \\ & \leq \int_0^t \|\Phi(H(s), \theta)\|_{L^2}^2 ds \\ & = -\mu(K, H(t)) \end{aligned}$$

Combining the above two inequalities, we have

$$(3.12) \quad (r^{-\frac{1}{2}}\|S(t)\|_{L^1} - V \log 2r)\|\Phi(H(t), \theta)\|_{L^2} \leq -V^{\frac{1}{2}}\mu(K, H(t)).$$

Claim Assume that the Higgs bundle (E, θ) is semi-stable, if $\mu(K, H(t)) \rightarrow -\infty$, then we must have

$$(3.13) \quad \lim_{t \rightarrow +\infty} \frac{-\mu(K, H(t))}{\|S(t)\|_{L^1}} = 0.$$

By the above claim and (3.12), we have $\|\Phi(H(t), \theta)\|_{L^2} \rightarrow 0$ as $t \rightarrow +\infty$. Combining the above two cases, we see that: if the Higgs bundle (E, θ) is semi-stable, then

$$(3.14) \quad \|\Phi(H(t), \theta)\|_{L^2} \rightarrow 0$$

as $t \rightarrow +\infty$.

Following Kobayashi's argument, let $u(x, t) = \int_M \chi(x, y, t-t_0) |\Phi(H(t_0), \theta)|^2 d\nu(y)$, where χ is the heat kernel. Using (2.5), we have

$$(3.15) \quad \left(\frac{\partial}{\partial t} - \Delta\right)(|\Phi(H(t), \theta)|_{H(t)}^2(x) - u(x, t)) \leq 0.$$

By the maximum principle,

$$\max_M (|\Phi(H(t), \theta)|_{H(t)}^2(x) - u(x, t)) \leq \max_M (|\Phi(H(t_0), \theta)|_{H(t_0)}^2(x) - u(x, t_0)) = 0.$$

Hence

$$(3.16) \quad \begin{aligned} \max_M |\Phi(H(t_0+1), \theta)|_H^2 & \leq \max_M u(x, t_0+1) \\ & = \int_M \chi(x, y, 1) |\Phi(H(t_0), \theta)|_{H(t_0)}^2 d\nu(y) \\ & \leq C \int_M |\Phi(H(t_0), \theta)|_{H(t_0)}^2 \frac{\omega^n}{n!} \rightarrow 0, \end{aligned}$$

as $t_0 \rightarrow +\infty$. So there exists an approximate H-E metric structure on semi-stable Higgs bundle (E, θ) .

To complete the proof of theorem 1, we only need to prove the above claim.

Proof of the Claim. We will follow Simpson's argument (Proposition 5.3 in [22]) to show that if the estimate does not hold, there is a sub Higgs-sheaf contradicting semi-stability.

Suppose the required estimate does not hold. We can find a positive constant C and a sequence $t_i \rightarrow +\infty$ such that

$$(3.17) \quad \frac{-\mu(K, H(t_i))}{\|S(t_i)\|_{L^1}} \geq C.$$

Set $u_i = l_i^{-1} S(t_i)$, where $l_i = \|S(t_i)\|_{L^1} \rightarrow +\infty$, then $\|u_i\|_{L^1} = 1$. By (2.11) and (3.6), we have $\text{tr} u_i = 0$ and $\|u_i\|_{L^\infty} \leq C_1$. Simpson proved that: $u_i \rightarrow u_\infty$ weakly in L_1^2 ; $\|u_\infty\|_{L^1} = 1$, and the eigenvalues of u_∞ are constant almost everywhere.

Let $\lambda_1 < \dots < \lambda_l$ denote the distinct eigenvalue of u_∞ . Since $tru_\infty = 0$ and $\|u_\infty\|_{L^1} = 1$, we must have $l \geq 2$. For any $1 \leq \alpha < l$, define function $P_\alpha : R \rightarrow R$ such that

$$(3.18) \quad P_\alpha = \begin{cases} 1, & x \leq \lambda_\alpha \\ 0, & x \geq \lambda_{\alpha+1} \end{cases}$$

Set $\pi_\alpha = P_\alpha(u_\infty)$, Simpson (p887 in [22]) proved that:

- (1) $\pi_\alpha \in L_1^2$;
- (2) $\pi_\alpha^2 = \pi_\alpha = \pi_\alpha^{*K}$;
- (3) $(Id - \pi_\alpha)\bar{\partial}\pi_\alpha = 0$;
- (4) $(Id - \pi_\alpha)[\theta, \pi_\alpha] = 0$.

By Uhlenbeck and Yau's regularity statement of L_1^2 -subbundle ([24]), π_α represent coherent torsion-free sub Higgs-sheaf E_α of (E, θ) . Set

$$(3.19) \quad \nu = \lambda_l \deg(E) - \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \deg(E_\alpha).$$

Since $u_\infty = \lambda_l Id - \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \pi_\alpha$ and $tru_\infty = 0$, we have

$$(3.20) \quad \lambda_l \text{rank}(E) - \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \text{rank}(E_\alpha) = 0,$$

then

$$(3.21) \quad \nu = \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \text{rank}(E_\alpha) \left(\frac{\deg(E)}{\text{rank}(E)} - \frac{\deg(E_\alpha)}{\text{rank}(E_\alpha)} \right).$$

On the other hand, by the Gauss-Codazzi equation, we have

$$(3.22) \quad \deg(E_\alpha) = \int_M \text{tr}(\pi_\alpha \sqrt{-1} \Lambda_\omega F_{K, \theta}) - |D_\theta'' \pi_\alpha|_K^2 \frac{\omega^n}{n!}.$$

Since $D_\theta''(\pi_\alpha) = D_\theta''(P_\alpha(u_\infty)) = dP_\alpha(u_\infty)(D_\theta''u_\infty)$, where the function $df : R^2 \rightarrow R$ defined by

$$df(x, y) = \frac{f(x) - f(y)}{x - y},$$

and which is taken as $\frac{df}{dx}$ if $x = y$. Then, we have

$$(3.23) \quad \begin{aligned} \nu &= \int_M \text{tr}((\lambda_l Id - \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) \pi_\alpha) \sqrt{-1} \Lambda_\omega F_{K, \theta}) \\ &\quad + \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) |D_\theta'' \pi_\alpha|^2 \\ &= \int_M \text{tr}(u_\infty \sqrt{-1} \Lambda_\omega F_{K, \theta}) \\ &\quad + \langle \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) (dP_\alpha)^2(u_\infty)(D_\theta''u_\infty), D_\theta''u_\infty \rangle_K \end{aligned}$$

If $\lambda_k \neq \lambda_l$, it easy to check that

$$(3.24) \quad \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) (dP_\alpha)^2(\lambda_k, \lambda_l) = |\lambda_k - \lambda_l|^{-1}$$

and

$$(3.25) \quad l_i \Psi(l_i \lambda_k, l_i \lambda_l) \rightarrow = \begin{cases} (\lambda_k - \lambda_l)^{-1}, & \lambda_k > \lambda_l \\ +\infty & \lambda_k \leq \lambda_l. \end{cases}$$

By (2.15), (3.24), (3.25) and (3.23), we have

$$\begin{aligned}
 -C &\geq \lim_{i \rightarrow \infty} \frac{\mu(K, H(t_i))}{\|S(t_i)\|_{L^1}} \\
 &= \lim_{i \rightarrow \infty} \int_M \text{tr}(u_i \sqrt{-1} \wedge_\omega F_{K, \theta}) + \langle l_i \Psi(l_i u_i)(D''_\theta u_i), D''_\theta u_i \rangle_K \\
 (3.26) \quad &\geq \int_M \text{tr}(u_\infty \sqrt{-1} \wedge_\omega F_{K, \theta}) \\
 &\quad + \langle \sum_{\alpha=1}^{l-1} (\lambda_{\alpha+1} - \lambda_\alpha) (dP_\alpha)^2(u_\infty)(D''_\theta u_\infty), D''_\theta u_\infty \rangle_K \\
 &= \nu.
 \end{aligned}$$

On the other hand, (3.21) and the semi-stability imply $\nu \geq 0$, so we get a contradiction. \square

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